GALOIS THEORY FOR WEAK HOPF ALGEBRAS

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Abstract. We develop Hopf-Galois theory for weak Hopf algebras, and recover analogs of classical results for Hopf algebras. Our methods are based on the recently introduced Galois theory for corings. We focus on the situatation where the weak Hopf algebra is a groupoid algebra or its dual. We obtain weak versions of strongly graded rings, and classical Galois extensions.

Introduction

Weak bialgebras and Hopf algebras are generalizations of ordinary bialgebras and Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and comultiplicativity of the unit are replaced by weaker axioms. Perhaps the easiest example of a weak Hopf algebra is a groupoid algebra; other examples are face algebras [15], quantum groupoids [17] and generalized Kac algebras [21]. A purely algebraic approach can be found in [2] and [3].

The aim of this note is to develop Galois theory for weak Hopf algebras. A possible strategy could be to try to adapt the methods from classical Hopf-Galois theory to the weak situation. This would be unnecessarily complicated, since a more powerful tool became available recently.

Corings and comodules over corings were introduced by Sweedler in [20]. The notion was revived recently by Brzeziński [5], who saw the importance of a remark made by Takeuchi, that Hopf modules and many of their generalizations can be viewed as examples of comodules over corings. Many applications came out of this idea, an overview is given in [6].

One of the beautiful applications is a reformulation of descent and Galois theory using the language of corings. This was initiated in [5], and continued by the first author [8] and Wisbauer [22]. An observation in [5] is that weak Hopf modules can be viewed as comodules over a coring, and this implies that the general theory of Galois corings can be applied to weak Hopf algebras. This is what we will do in Section 2.

In Section 3, we look at the special case where the weak Hopf algebra is a groupoid algebra. This leads us to the notions of groupoid graded algebras and modules, and the following generalization of a result of Ulbrich: a groupoid graded algebra is Galois if and only if it is strongly graded.

In Section 4, we look at the dual situation, where the weak Hopf algebra

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is the dual of a finite groupoid algebra. Then we have to deal with finite groupoid actions, as we have to deal with finite group actions in classical Galois theory. The results in this Section are similar to the ones obtained in [10], where partial Galois theory (see [14]) is studied using the language of corings. Both are generalizations of classical Galois theory. In partial Galois theory, we look at group actions on algebra A that are not everywhere defined, and in weak Galois theory, we look at groupoid actions. In both cases, we have a coring that is a direct factor of $|G|$ copies of A, and the left dual of the coring is a Frobenius extension of A.

1. Preliminary results

1.1. **Galois corings.** Let A be a ring. An A-coring $\mathcal{C}$ is a coalgebra in the category ${}_A\mathcal{M}_A$ of A-bimodules. Thus an A-coring is a triple $\mathcal{C} = (\mathcal{C}, \Delta_\mathcal{C}, \varepsilon_\mathcal{C})$, where $\mathcal{C}$ is an A-bimodule, and $\Delta_\mathcal{C} :\ \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon_\mathcal{C} :\ \mathcal{C} \to A$ are A-bimodule maps such that

$$(\Delta_\mathcal{C} \otimes_A \mathcal{C}) \circ \Delta_\mathcal{C} = (\mathcal{C} \otimes_A \Delta_\mathcal{C}) \circ \Delta_\mathcal{C}, \tag{1}$$

and

$$(\mathcal{C} \otimes_A \varepsilon_\mathcal{C}) \circ \Delta_\mathcal{C} = (\varepsilon_\mathcal{C} \otimes_A \mathcal{C}) \circ \Delta_\mathcal{C} = \mathcal{C}. \tag{2}$$

We use the Sweedler-Heyneman notation for the comultiplication:

$$\Delta_\mathcal{C}(c) = c_{(1)} \otimes_A c_{(2)}.$$

A right $\mathcal{C}$-comodule $M = (M, \rho)$ consists of a right A-module M together with a right A-linear map $\rho :\ M \to M \otimes_A \mathcal{C}$ such that:

$$(\rho \otimes_A \mathcal{C}) \circ \rho = (M \otimes_A \Delta_\mathcal{C}) \circ \rho, \tag{3}$$

and

$$(M \otimes_A \varepsilon_\mathcal{C}) \circ \rho = M. \tag{4}$$

We then say that $\mathcal{C}$ coacts from the right on M, and we denote

$$\rho(m) = m_{[0]} \otimes_A m_{[1]}.$$

A right A-linear map $f :\ M \to N$ between two right $\mathcal{C}$-comodules M and N is called right $\mathcal{C}$-colinear if $\rho(f(m)) = f(m_{[0]}) \otimes m_{[1]}$, for all $m \in M$. The category of right $\mathcal{C}$-comodules and $\mathcal{C}$-colinear maps is denoted by $\mathcal{M}^\mathcal{C}$.
$x \in \mathcal{C}$ is called grouplike if $\Delta_\mathcal{C}(x) = x \otimes x$ and $\varepsilon_\mathcal{C}(x) = 1$. Grouplike elements of $\mathcal{C}$ correspond bijectively to right $\mathcal{C}$-coactions on A: if A is grouplike, then we have the following right $\mathcal{C}$-coaction ρ on A: $\rho(a) = xa$.
Let $(\mathcal{C}, x)$ be a coring with a fixed grouplike element. For $M \in \mathcal{M}^\mathcal{C}$, we call

$$M^{\mathrm{co}\mathcal{C}} = \{m \in M \mid \rho(m) = m \otimes_A x\}$$

the submodule of coinvariants of M. Observe that

$$A^{\mathrm{co}\mathcal{C}} = \{b \in A \mid bx = xb\}$$

is a subring of A. Let $i:\ B\to A$ be a ring morphism. i factorizes through $A^{\mathrm{co}\mathcal{C}}$ if and only if

$$x\in G(\mathcal{C})^B=\{x\in G(\mathcal{C})\mid xb=bx,\ \text{for all } b\in B\}.$$

We then have a pair of adjoint functors (F,G), respectively between the categories $\mathcal{M}_B$ and $\mathcal{M}^{\mathcal{C}}$ and the categories ${}_B\mathcal{M}$ and ${}^{\mathcal{C}}\mathcal{M}$. For $N\in\mathcal{M}_B$ and $M\in\mathcal{M}^{\mathcal{C}}$,

$$F(N)=N\otimes_B A \text{ and } G(M)=M^{\mathrm{co}\mathcal{C}}.$$

The unit and counit of the adjunction are

$$\nu_N:\ N\to(N\otimes_B A)^{\mathrm{co}\mathcal{C}},\ \nu_N(n)=n\otimes_B 1;$$

$$\zeta_M:\ M^{\mathrm{co}\mathcal{C}}\otimes_B A\to M,\ \zeta_M(m\otimes_B a)=ma.$$

We have a similar pair of adjoint functors (F',G') between the categories of left B-modules and left $\mathcal{C}$-comodules.
Let $i:\ B\to A$ be a morphism of rings. The associated canonical coring is $\mathcal{D}=A\otimes_B A$, with comultiplication and counit given by the formulas

$$\Delta_{\mathcal{D}}:\ \mathcal{D}\to\mathcal{D}\otimes_A\mathcal{D}\cong A\otimes_B A\otimes_B A,\ \Delta_{\mathcal{D}}(a\otimes_B a')=a\otimes_B 1\otimes_B a'$$

and

$$\varepsilon_{\mathcal{D}}:\ \mathcal{D}=A\otimes_B A\to A,\ \varepsilon_{\mathcal{D}}(a\otimes_B a')=aa'.$$

If $i:\ B\to A$ is pure as a morphism of left and right B-modules, then the categories $\mathcal{M}_B$ and $\mathcal{M}^{\mathcal{D}}$ are equivalent.
Let $(\mathcal{C},x)$ be a coring with a fixed grouplike element, and $i:\ B\to A^{\mathrm{co}\mathcal{C}}$ a ring morphism. We then have a morphism of corings

$$\mathrm{can}:\ \mathcal{D}=A\otimes_B A\to\mathcal{C},\ \mathrm{can}(a\otimes_B a')=axa'.$$

If F is fully faithful, then $B\cong A^{\mathrm{co}\mathcal{C}}$; if G is fully faithful, then can is an isomorphism. $(\mathcal{C},x)$ is called a Galois coring if $\mathrm{can}:\ A\otimes_{A^{\mathrm{co}\mathcal{C}}}A\to\mathcal{C}$ is bijective. From [8], we recall the following results.

Theorem 1.1. *Let $(\mathcal{C},x)$ be an A-coring with fixed grouplike element, and $T=A^{\mathrm{co}\mathcal{C}}$. Then the following statements are equivalent.*

(1) *$(\mathcal{C},x)$ is Galois and A is faithfully flat as a left T-module;*
(2) *(F,G) is an equivalence and A is flat as a left T-module.*

Let $(\mathcal{C},x)$ be a coring with a fixed grouplike element, and take $T=A^{\mathrm{co}\mathcal{C}}$. Then ${}^*\mathcal{C}={}_A\mathrm{Hom}(\mathcal{C},A)$ is a ring, with multiplication given by

$$(f\#g)(c)=g(c_{(1)}f(c_{(2)})). \tag{5}$$

We have a morphism of rings $j:\ A\to{}^*\mathcal{C}$, given by

$$j(a)(c)=\varepsilon_{\mathcal{C}}(c)a.$$

This makes ${}^*\mathcal{C}$ into an A-bimodule, via the formula

$$(afb)(c)=f(ca)b.$$

Consider the left dual of the canonical map:

$${}^*\mathrm{can}:\ {}^*\mathcal{C}\to{}^*\mathcal{D}\cong{}_T\mathrm{End}(A)^{\mathrm{op}},\ {}^*\mathrm{can}(f)(a)=f(xa).$$

We then have the following result.

Proposition 1.2. *If $(\mathcal{C}, x)$ is Galois, then* ${}^*\mathrm{can}$ *is an isomorphism. The converse property holds if $\mathcal{C}$ and A are finitely generated projective, respectively as a left A-module, and a left T-module.*

Let $Q = \{q \in {}^*\mathcal{C} \mid c_{(1)}q(c_{(2)}) = q(c)x,\ \text{for all } c \in \mathcal{C}\}$. A straightforward computation shows that Q is a $({}^*\mathcal{C}, T)$-bimodule. Also A is a left $(T, {}^*\mathcal{C})$-bimodule; the right ${}^*\mathcal{C}$-action is induced by the right $\mathcal{C}$-coaction: $a \cdot f = f(xa)$. Now consider the maps

$$\tau: \quad A \otimes_{{}^*\mathcal{C}} Q \to T,\ \tau(a \otimes_{{}^*\mathcal{C}} q) = q(xa); \tag{6}$$

$$\mu: \quad Q \otimes_T A \to {}^*\mathcal{C},\ \mu(q \otimes_T a) = q\#i(a). \tag{7}$$

With this notation, we have the following property (see [12]).

Proposition 1.3. *$(T, {}^*\mathcal{C}, A, Q, \tau, \mu)$ is a Morita context. The map τ is surjective if and only if there exists $q \in Q$ such that $q(x) = 1$.*

We also have (see [8]):

Theorem 1.4. *Let $(\mathcal{C}, x)$ be a coring with fixed grouplike element, and assume that $\mathcal{C}$ is a left A-progenerator. We take a subring B of $T = A^{\mathrm{co}\mathcal{C}}$, and consider the map*

$$\mathrm{can}:\ \mathcal{D} = A \otimes_B A \to \mathcal{C},\ \mathrm{can}(a \otimes_T a') = axa'$$

Then the following statements are equivalent:

(1)
- can *is an isomorphism;*
- *A is faithfully flat as a left B-module.*

(2)
- ${}^*\mathrm{can}$ *is an isomorphism;*
- *A is a left B-progenerator.*

(3)
- *$B = T$;*
- *the Morita context $(B, {}^*\mathcal{C}, A, Q, \tau, \mu)$ is strict.*

(4)
- *$B = T$;*
- *(F, G) is an equivalence of categories.*

1.2. **Weak Hopf algebras.** Let k be a commutative ring. Recall that a weak k-bialgebra is a k-module with a k-algebra structure (μ, η) and a k-coalgebra structure (Δ, ε) such that

$$\Delta(hk) = \Delta(h)\Delta(k),$$

for all $h, k \in H$, and

$$\Delta^2(1) = 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')}, \tag{8}$$

$$\varepsilon(hkl) = \varepsilon(hk_{(1)})\varepsilon(k_{(2)}l) = \varepsilon(hk_{(2)})\varepsilon(k_{(1)}l), \tag{9}$$

for all $h, k, l \in H$. We use the Sweedler-Heyneman notation for the comultiplication, namely

$$\Delta(h) = h_{(1)} \otimes h_{(2)} = h_{(1')} \otimes h_{(2')}.$$

If H is a weak bialgebra, then we have idempotent maps $\Pi^L, \Pi^R, \overline{\Pi}^L, \overline{\Pi}^R :$ $H \to H$ given by

$$\begin{aligned}
\Pi^L(h) &= \varepsilon(1_{(1)}h)1_{(2)}\\
\Pi^R(h) &= 1_{(1)}\varepsilon(h1_{(2)})\\
\overline{\Pi}^L(h) &= 1_{(1)}\varepsilon(1_{(2)}h)\\
\overline{\Pi}^R(h) &= \varepsilon(h1_{(1)})1_{(2)}
\end{aligned}$$

We have that $H^L = \mathrm{Im}\,(\Pi^L) = \mathrm{Im}\,(\overline{\Pi}^R)$ and $H^R = \mathrm{Im}\,(\Pi^R) = \mathrm{Im}\,(\overline{\Pi}^L)$.
Let H be a weak bialgebra. Recall from [1] and [9] that a right H-comodule algebra A is a k-algebra with a right H-comodule structure ρ such that $\rho(a)\rho(b) = \rho(ab)$, for all $a, b \in A$, and such that the following equivalent statements hold (see [9, Prop. 4.10]):

$$\rho^2(1) = \sum 1_{[0]} \otimes 1_{[1]}1_{(1)} \otimes 1_{(2)} \tag{10}$$

$$\rho^2(1) = \sum 1_{[0]} \otimes 1_{(1)}1_{[1]} \otimes 1_{(2)} \tag{11}$$

$$\sum a_{[0]} \otimes \overline{\Pi}^R(a_{[1]}) = \sum a1_{[0]} \otimes 1_{[1]} \tag{12}$$

$$\sum a_{[0]} \otimes \Pi^L(a_{[1]}) = \sum 1_{[0]}a \otimes 1_{[1]} \tag{13}$$

$$\sum 1_{[0]} \otimes \overline{\Pi}^R(1_{[1]}) = \rho^r(1) \tag{14}$$

$$\sum 1_{[0]} \otimes \Pi^L(1_{[1]}) = \rho^r(1) \tag{15}$$

$$\rho^r(1) \in A \otimes H^L \tag{16}$$

Lemma 1.5. *Let H be a weak bialgebra, and A a right H-comodule algebra. Then*

$$\varepsilon(h_{(1)}1_{[1]})1_{[0]} \otimes h_{(2)} = 1_{[0]} \otimes h1_{[1]} \tag{17}$$

$$\varepsilon(h1_{[1]})1_{[0]}a = \varepsilon(ha_{[1]})a_{[0]} \tag{18}$$

for all $h \in H$ and $a \in A$.

Proof. This is a special case of [9, Theorem 4.14]. Details are as follows.

$$\begin{aligned}
\varepsilon(h_{(1)}1_{[1]})1_{[0]} \otimes h_{(2)} &= \varepsilon(h_{(1)}1_{(1)}1_{[1]})1_{[0]} \otimes h_{(2)}1_{(2)}\\
&= \varepsilon(h_{(1)}1_{[1]})1_{[0]} \otimes h_{(2)}1_{[2]} = 1_{[0]} \otimes h1_{[1]};\\
\varepsilon(h1_{[1]})1_{[0]}a &= \varepsilon(h1_{[2]})\varepsilon(1_{[1]}a_{[1]})1_{[0]}a_{[0]}\\
&= \varepsilon(h1_{[1]}a_{[1]})1_{[0]}a_{[0]} = \varepsilon(ha_{[1]})a_{[0]}.
\end{aligned}$$

□

A weak Hopf algebra is a weak bialgebra together with a map $S:\ H \to H$, called the antipode, satisfying the following conditions, for all $h \in H$:

$$h_{(1)}S(h_{(2)}) = \Pi^L(h);\ \ S(h_{(1)})h_{(2)} = \Pi^R(h); \tag{19}$$

$$S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h). \tag{20}$$

2. Weak Hopf-Galois extensions

Let H be a weak bialgebra, and A a right H-comodule algebra. We then have a projection

$$g:\ A\otimes H\to A\otimes H,\ g(a\otimes h)=a1_{[0]}\otimes h1_{[1]}.$$

Observe that, for any $a\in A$,

$$\rho(a)=a_{[0]}\otimes a_{[1]}=a_{[0]}1_{[0]}\otimes a_{[1]}1_{[1]}=g(\rho(a))\in \mathrm{Im}\,(g).$$

Lemma 2.1. *Let H be a weak bialgebra. Then $\mathcal{C}=\mathrm{Im}\,(g)$ is an A-coring, with structure maps*

$$\begin{aligned} b'(a1_{[0]}\otimes h1_{[1]})b &= b'ab_{[0]}\otimes hb_{[1]};\\ \Delta_{\mathcal{C}}(1_{[0]}\otimes h1_{[1]}) &= (1_{[0]}\otimes h_{(1)}1_{[1]})\otimes_A(1\otimes h_{(2)});\\ \varepsilon_{\mathcal{C}}(1_{[0]}\otimes h1_{[1]}) &= 1_{[0]}\varepsilon(h1_{[1]}). \end{aligned}$$

$\rho(1)=1_{[0]}\otimes 1_{[1]}$ is a grouplike element in $\mathcal{C}$.

Proof. This is a special case of a result in [4, Prop. 2.3]. The fact that $\Delta_{\mathcal{C}}(1_{[0]}\otimes h1_{[1]})\in\mathcal{C}\otimes_A\mathcal{C}$ follows from

$$\Delta_{\mathcal{C}}(1_{[0]}\otimes h1_{[1]})=(1_{[0]}\otimes h_{(1)}1_{[1]})\otimes_A((1_{[0']}\otimes h_{(2)}1_{[1']}). \tag{21}$$

Indeed,

$$\begin{aligned} &(1_{[0]}\otimes h_{(1)}1_{[1]})\otimes_A((1_{[0']}\otimes h_{(2)}1_{[1']})\\ &\quad= (1_{[0]}1_{[0']}\otimes h_{(1)}1_{[1]}1_{[1']})\otimes_A(1\otimes h_{(2)}1_{[2']})\\ &\quad= (1_{[0]}1_{[0']}\otimes h_{(1)}1_{[1]}1_{[1']}1_{(1)})\otimes_A(1\otimes h_{(2)}1_{(2)})\\ &\quad= (1_{[0]}\otimes h_{(1)}1_{[1]}1_{(1)})\otimes_A(1\otimes h_{(2)}1_{(2)})\\ &\quad= (1_{[0]}\otimes h_{(1)}1_{(1)}1_{[1]})\otimes_A(1\otimes h_{(2)}1_{(2)})=\rho(a). \end{aligned}$$

Taking $h=1$ in (21) and (21), we see that $\varepsilon_{\mathcal{C}}(\rho(1))=1$ and $\Delta_{\mathcal{C}}(\rho(1))=\rho(1)\otimes_A\rho(1)$. □

We call (M,ρ) a relative right (A,H)-Hopf module if M is a right A-module, (M,ρ) is a right H-comodule, and

$$\rho(ma)=m_{[0]}a_{[0]}\otimes m_{[1]}a_{[1]}, \tag{22}$$

for all $m\in M$ and $a\in A$. $\mathcal{M}_A^H$ is the category of relative Hopf modules and right A-linear H-colinear maps.
The following result is a special case of [4, Prop. 2.3]. Since it is essential in what follows, we give a sketch of proof.

Proposition 2.2. *The category $\mathcal{M}_A^H$ of relative Hopf modules is isomorphic to the category $\mathcal{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules over the coring $\mathcal{C}=\mathrm{Im}\,g$.*

Proof. Take $(M,\tilde{\rho})\in\mathcal{M}^{\mathcal{C}}$. Let ρ be the composition

$$\rho:\ M\xrightarrow{\tilde{\rho}} M\otimes_A\mathcal{C}\longrightarrow M\otimes_A(A\otimes H)\xrightarrow{\cong} M\otimes H.$$

If

$$\tilde{\rho}(m)=\sum_i m_i\otimes_A a_i1_{[0]}\otimes h_i1_{[1]},$$

then

$$\rho(m)=m_{[0]}\otimes m_{[1]}=\sum_i m_ia_i1_{[0]}\otimes h_i1_{[1]}.$$

Then $\tilde{\rho}$ is determined by ρ, since

$$\begin{aligned}\tilde{\rho}(m) &= \sum_i m_i\otimes_A a_i1_{[0]}\otimes h_i1_{[1]}\\ &= \sum_i m_i\otimes_A a_i1_{[0]}1_{[0']}\otimes h_i1_{[1]}1_{[0']}\\ &= \sum_i m_ia_i1_{[0]}\otimes_A 1_{[0']}\otimes h_i1_{[1]}1_{[0']}\\ &= m_{[0]}\otimes_A 1_{[0]}\otimes m_{[1]}1_{[1]}.\end{aligned}\tag{23}$$

From the fact that $\tilde{\rho}$ is right A-linear, it follows that

$$\begin{aligned}\tilde{\rho}(ma)=\tilde{\rho}(m)a &= m_{[0]}\otimes_A(1_{[0]}\otimes m_{[1]}1_{[1]})a\\ &= m_{[0]}\otimes_A a_{[0]}\otimes m_{[1]}a_{[1]},\end{aligned}$$

so

$$\rho(ma)=m_{[0]}a_{[0]}\otimes m_{[1]}a_{[1]}.$$

From the fact that $(M\otimes_A\varepsilon_{\mathcal{C}})(\tilde{\rho}(m))=m$ and (23), it follows that

$$m=m_{[0]}1_{[0]}\otimes\varepsilon(m_{[1]}1_{[1]})=m_{[0]}\otimes\varepsilon(m_{[1]}).$$

For all $m\in M$, we have the following equality in $M\otimes_A(A\otimes H)\otimes_A(A\otimes H)$;

$$(M\otimes_A\Delta_{\mathcal{C}})(\tilde{\rho}(m))=(\tilde{\rho}\otimes\mathcal{C})(\tilde{\rho}(m))\tag{24}$$

The left hand side of (24) is

$$m_{[0][0]}\otimes_A(1_{[0']}\otimes m_{[0][1]}1_{[1']})\otimes_A(1_{[0]}\otimes m_{[1]}1_{[1]}).$$

The image in $M\otimes_A(A\otimes H\otimes H)$ is

$$\begin{aligned}&m_{[0][0]}\otimes_A(1_{[0']}1_{[0]}\otimes m_{[0][1]}1_{[1']}1_{[1]}1_{(1)}\otimes m_{[1]}1_{(2)})\\ &\quad= m_{[0][0]}\otimes_A(1_{[0]}\otimes m_{[0][1]}1_{[1]}1_{(1)}\otimes m_{[1]}1_{(2)}),\end{aligned}$$

and in $M\otimes H\otimes H$:

$$\begin{aligned}&m_{[0][0]}1_{[0]}\otimes m_{[0][1]}1_{[1]}1_{(1)}\otimes m_{[1]}1_{(2)}\\ &\quad= m_{[0][0]}1_{[0]}\otimes m_{[0][1]}1_{[1]}\otimes m_{[1]}1_{[2]}\\ &\quad= m_{[0][0]}1_{[0][0]}\otimes m_{[0][1]}1_{[0][1]}\otimes m_{[1]}1_{[1]}\\ &\quad= m_{[0][0]}\otimes m_{[0][1]}\otimes m_{[1]}.\end{aligned}$$

The right hand side of (24) is

$$m_{[0]} \otimes_A (1_{[0]} \otimes m_{[1](1)} 1_{[1]}) \otimes_A (1_{[0']} \otimes m_{[1](2)} 1_{[1']}).$$

The image in $M \otimes_A (A \otimes H \otimes H)$ is

$$\begin{aligned} & m_{[0]} \otimes_A (1_{[0]} 1_{[0']} \otimes m_{[1](1)} 1_{[1]} 1_{[1']} 1_{(1)} \otimes m_{[1](2)} 1_{(2)}) \\ &= m_{[0]} \otimes_A (1_{[0]} \otimes m_{[1](1)} 1_{[1]} 1_{(1)} \otimes m_{[1](2)} 1_{(2)}), \end{aligned}$$

and in $M \otimes H \otimes H$:

$$\begin{aligned} & m_{[0]} 1_{[0]} \otimes m_{[1](1)} 1_{[1]} 1_{(1)} \otimes m_{[1](2)} 1_{(2)} \\ &= m_{[0]} 1_{[0]} \otimes m_{[1](1)} 1_{[1]} \otimes m_{[1](2)} 1_{[2]} \\ &= m_{[0]} 1_{[0]} \otimes m_{[1](1)} 1_{[1](1)} \otimes m_{[1](2)} 1_{[1](2)} \\ &= m_{[0]} \otimes m_{[1](1)} \otimes m_{[1](2)}. \end{aligned}$$

It follows that

$$m_{[0][0]} \otimes m_{[0][1]} \otimes m_{[1]} = m_{[0]} \otimes m_{[1](1)} \otimes m_{[1](2)},$$

and ρ is coassociative. Conversely, given a relative Hopf module (M, ρ), we define $\tilde{\rho} :\ M \to M \otimes_A \mathcal{C}$ using (23). Straightforward computations show that $(M, \tilde{\rho}) \in \mathcal{M}^{\mathcal{C}}$. $\square$

Let (M, ρ) be a relative Hopf module, and $(M, \tilde{\rho})$ the corresponding $\mathcal{C}$-comodule. Then $m \in M^{\mathrm{co}\mathcal{C}}$ if and only if

$$\tilde{\rho}(m) = m \otimes_A 1_{[0]} \otimes 1_{[1]}$$

if and only if

$$\rho(m) = m 1_{[0]} \otimes 1_{[1]}.$$

We conclude that $M^{\mathrm{co}\mathcal{C}} = M^{\mathrm{co}H}$, which is by definition

$$M^{\mathrm{co}H} = \{m \in M \mid \rho(m) = m 1_{[0]} \otimes 1_{[1]}\}.$$

The grouplike element $\rho(1)$ induces a right $\mathcal{C}$-coaction $\tilde{\rho}$ on A:

$$\tilde{\rho}(a) = 1 \otimes_A (1_{[0]} \otimes 1_{[1]}) a = 1 \otimes_A (a_{[0]} \otimes a_{[1]}).$$

The corresponding H-coaction on A is the original ρ:

$$\rho(a) = a_{[0]} \otimes a_{[1]}.$$

Observe that

$$T = A^{\mathrm{co}H} = \{a \in A \mid a 1_{[0]} \otimes 1_{[1]}\}.$$

Let $i :\ B \to T$ be a ring morphism. We have seen in Section 1.1 that we have a pair of adjoint functors (F, G):

$$F :\ \mathcal{M}_B \to \mathcal{M}_A^H,\ F(N) = N \otimes_B A;$$

$$G :\ \mathcal{M}_A^H \to \mathcal{M}_B,\ G(N) = N^{\mathrm{co}H}.$$

$F(N) = N \otimes_B A$ is a relative Hopf module via

$$\rho(n \otimes_B a) = n \otimes_B a_{[0]} \otimes a_{[1]}.$$

The canonical map can : $A \otimes_B A \to \mathcal{C}$ takes the form

$$\mathrm{can}(a \otimes_B b) = a(1_{[0]} \otimes 1_{[1]})b = ab_{[0]} \otimes b_{[1]}. \tag{25}$$

We call A a weak H-Galois extension of $A^{\mathrm{co}H}$ if $(\mathcal{C}, \rho(1))$ is a Galois coring, that is, can : $A \otimes_{A^{\mathrm{co}H}} A \to \mathcal{C}$ is an isomorphism. From Theorem 1.1, we immediately have the following result.

Proposition 2.3. *Let H be a weak bialgebra, and A a right H-comodule algebra. Then the following assertions are equivalent*

1. *A is A a weak H-Galois extension of T and faithfully flat as a left B-module;*
2. *(F, G) is an equivalence and A is flat as a left T-module.*

Our next aim is to compute ${}^*\mathcal{C}$.

Proposition 2.4. *Let H be a weak bialgebra, A a right H-comodule algebra, and $\mathcal{C} = \mathrm{Im}\,(g)$ the A-coring that we introduced above. Then*

$$ {}^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C}, A) \cong \underline{\mathrm{Hom}}(H, A)$$
$$\{f \in \mathrm{Hom}(H, A) \mid f(h) =: 1_{[0]}f(h1_{[1]}),\ \text{for all } h \in H\},$$

with multiplication rule

$$(f\#g)(h) = f(h_{(2)})_{[0]}g(h_{(1)}f(h_{(2)})_{[1]}). \tag{26}$$

Proof. We define $\alpha :\ {}^*\mathcal{C} \to \underline{\mathrm{Hom}}(H, A)$ by

$$\alpha(\varphi)(h) = \varphi(1_{[0]} \otimes h1_{[1]}).$$

$\alpha(\varphi) = f \in \underline{\mathrm{Hom}}(H, A)$ since

$$\begin{aligned} 0 &= \alpha(\varphi)((1 \otimes h)(\rho(1) - \rho(1)^2)) \\ &= \alpha(\varphi)(1_{[0]} \otimes h1_{[1]}) - \alpha(\varphi)(1_{[0]}1_{[0']} \otimes h1_{[1]}1_{[1']}) \\ &= f(h) - 1_{[0]}f(h1_{[1]}). \end{aligned}$$

Now we define $\beta :\ \underline{\mathrm{Hom}}(H, A) \to {}^*\mathcal{C}$ by

$$\beta(f)(a1_{[0]} \otimes h1_{[1]}) = af(h).$$

We have to show that β is well-defined. Assume that $\sum_i a_i1_{[0]} \otimes h_i1_{[1]} = 0$ in $A \otimes H$. Then

$$\sum_i a_i \otimes h_i = \sum_i a_i \otimes h_i - \sum_i a_i1_{[0]} \otimes h_i1_{[1]},$$

hence

$$\sum_i a_if(h_i) = \sum_i a_if(h_i) - \sum_i a_i1_{[0]}f(h_i1_{[1]}) = 0.$$

A straightforward verification shows that α is the inverse of β. Using α and β, we can transport the multiplication on ${}^*\mathcal{C}$ to $\underline{\mathrm{Hom}}(H,A)$. This gives

$$\begin{aligned}(f\#g)(h) = (\varphi\#\psi)(1_{[0]}\otimes h1_{[1]})\\ &= \psi((1_{[0]}\otimes h_{(1)}1_{[1]})\varphi(1_{[0']}\otimes h_{(2)}1_{[1']})\\ &= \psi((1_{[0]}\otimes h_{(1)}1_{[1]})f(h_{(2)})\\ &= \psi(f(h_{(2)})_{[0]}\otimes h_{(1)}f(h_{(2)})_{[1]})\\ &= f(h_{(2)})_{[0]}g(h_{(1)}f(h_{(2)})_{[1]}),\end{aligned}$$

as needed. □

The dual of the canonical map ${}^*\mathrm{can}:\ {}^*\mathcal{C}\to {}_A\mathrm{Hom}(A\otimes_B A,A)$ is given by

$$ {}^*\mathrm{can}(\varphi)(b\otimes_A a)=\varphi(\mathrm{can}(b\otimes_A a))=\varphi(ba_{[0]}\otimes a_{[1]}),$$

and transports to ${}^*\mathrm{can}:\ \underline{\mathrm{Hom}}(H,A)\to {}_B\mathrm{End}(A)^{\mathrm{op}}$, given by

$$ {}^*\mathrm{can}(f)(a)={}^*\mathrm{can}(\varphi)(1\otimes_B a)=a_{[0]}f(a_{[1]}). \tag{27}$$

Remarks 2.5. 1) We have a projection $p:\ \mathrm{Hom}(H,A)\to\underline{\mathrm{Hom}}(H,A)$, $p(f)=\tilde{f}$, given by

$$\tilde{f}(h)=1_{[0]}f(1_{[1]}).$$

2) The unit element of $\underline{\mathrm{Hom}}(H,A)$ is not ε, but $\tilde{\varepsilon}=\alpha(\varepsilon_{\mathcal{C}})$. This can be verified directly as follows:

$$\begin{aligned}(\tilde{\varepsilon}\#g)(h) = 1_{[0]}\varepsilon(h_{(2)}1_{[2]})g(h_{(1)}1_{[1]})\\ &= 1_{[0]}\varepsilon(h_{(2)}1_{(2)})g(h_{(1)}1_{(1)}1_{[1]})=1_{[0]}\varepsilon(h_{(2)}g(h_{(1)}1_{[1]})\\ &= 1_{[0]}g(h1_{[1]})=g(h);\\ (f\#\tilde{\varepsilon})(h) = f(h_{(2)})_{[0]}\tilde{\varepsilon}(h_{(1)}f(h_{(2)})_{[1]})\\ &= f(h_{(2)})_{[0]}1_{[0']}\varepsilon(h_{(1)}f(h_{(2)})_{[1]}1_{[0']})=f(h_{(2)})_{[0]}\varepsilon(h_{(1)}f(h_{(2)})_{[1]})\\ (18)\quad &= 1_{[0]}\varepsilon(h_{(1)}1_{[1]})f(h_{(2)})\\ (17)\quad &= 1_{[0]}f(h1_{[1]})=f(h).\end{aligned}$$

3) ${}^*\mathcal{C}$ is an A-bimodule, hence $\mathrm{Hom}(H,A)$ is also an A-bimodule. The structure is given by the formula

$$(afb)(h)=a_{[0]}f(ha_{[1]})b. \tag{28}$$

Let us next compute the Morita context $(T,\underline{\mathrm{Hom}}(H,A),A,Q,\tau,\mu)$ from Proposition 1.3. We have a ring morphism $j:\ A\xrightarrow{i}{}^*\mathcal{C}\xrightarrow{\alpha}\underline{\mathrm{Hom}}(H,A)$, given by

$$j(a)(h)=1_{[0]}\varepsilon(h1_{[1]})a.$$

Take $\varphi\in Q=\{\varphi\in{}^*\mathcal{C}\mid c_{(1)}\varphi(c_{(2)})=\varphi(c)\rho(1),\ \text{for all } c\in\mathcal{C}\}$, and let $\alpha(\varphi)=q$. Then

$$c_{(1)}\varphi(c_{(2)})=(1_{[0]}\otimes h_{(1)}1_{[1]})f(h_{(2)})=f(h_{(2)})_{[0]}\otimes h_{(1)}f(h_{(2)})_{[1]},$$

and

$$\varphi(c)\rho(1)=f(h)1_{[0]}\otimes 1_{[1]},$$

hence Q is the subset of $\underline{\mathrm{Hom}}(H,A)$ consisting of maps $f:\ H\to A$ satisfying

$$f(h_{(2)})_{[0]}\otimes h_{(1)}f(h_{(2)})_{[1]}=f(h)1_{[0]}\otimes 1_{[1]}, \tag{29}$$

for all $h\in H$. Q is an $(\underline{\mathrm{Hom}}(H,A),T)$-bimodule: $f\cdot q\cdot a=f\#q\#j(a)$, for all $f\in\underline{\mathrm{Hom}}(H,A)$, $q\in Q$ and $a\in A$.
A is a $(T,\underline{\mathrm{Hom}}(H,A))$-bimodule; the left T-action is given by left multiplication, and the right $\underline{\mathrm{Hom}}(H,A)$)-action is given by $a\cdot q=a_{[0]}q(a_{[1]})$. The connecting maps

$$\tau:\ A\otimes_{\underline{\mathrm{Hom}}(H,A)}Q\to T \text{ and } \mu:\ Q\otimes_T A\to\underline{\mathrm{Hom}}(H,A)$$

are given by the formulas

$$\tau(a\otimes q)=a_{[0]}q(a_{[1]});$$

$$\mu(q\otimes a)=q\#j(a),\ \text{that is}\ \mu(q\otimes a)(h)=q(h)a.$$

It follows from Proposition 1.2 that τ is surjective if and only if there exists $q\in Q$ such that $1_{[0]}q(1_{[1]})=1$. Theorem 1.4 takes the following form.

Theorem 2.6. *Let H be a finitely generated projective weak bialgebra, and A a right H-comodule algebra. Take a subring B of $T=A^{\mathrm{co}H}$. Then, with notation as above, the following assertions are equivalent.*

(1) • can *is an isomorphism;*
  • *A is faithfully flat as a left B-module.*

(2) • *can *is an isomorphism;*
  • *A is a left B-progenerator.*

(3) • *$B=T$;*
  • *the Morita context $(B,\underline{\mathrm{Hom}}(H,A),A,Q,\tau,\mu)$ is strict.*

(4) • *$B=T$;*
  • *(F,G) is an equivalence of categories.*

Proposition 2.7. *Let H be a weak Hopf algebra. Then H is a weak H-Galois extension of $H^{\mathrm{co}H}=H^L$.*

Proof. Let us first show that $H^{\mathrm{co}H}=H^L$. If $h\in H^{\mathrm{co}H}$, then $\Delta(h)=h1_{(1)}\otimes 1_{(2)}$, hence $h=\varepsilon(h1_{(1)})1_{(2)}=\overline{\Pi}^R(h)\in H^L$.
Conversely, if $h\in H^L$, then $h=\overline{\Pi}^R(h)=\varepsilon(h1_{(1)})1_{(2)}$, so

$$\begin{aligned}\Delta(h)&=\varepsilon(h1_{(1)})1_{(2)}\otimes 1_{(3)}\\ &=\ \varepsilon(h1_{(1)})1_{(2)}1_{(1')}\otimes 1_{(2')}=h1_{(1)}\otimes 1_{(2)},\end{aligned}$$

hence $h\in H^{\mathrm{co}H}$.
Let us next show that can is invertible, with inverse

$$\mathrm{can}^{-1}(1_{(1)}\otimes h1_{(2)})=1_{(1)}S(h_{(1)}1_{(2)}\otimes_{H^L}h_{(2)}1_{(3)}.$$

$$\begin{aligned}
\mathrm{can}^{-1}(\mathrm{can}(h\otimes k)) &= \mathrm{can}^{-1}(hk_{(1)}\otimes k_{(2)}) = hk_{(1)}S(k_{(2)})\otimes_{H^L} k_{(3)}\\
&= h\Pi^L(k_{(1)})\otimes_{H^L} k_{(2)} = h\otimes_{H^L}\Pi^L(k_{(1)})k_{(2)}\\
&= h\otimes_{H^L}\varepsilon(1_{(1)}k_{(1)})1_{(2)}k_{(2)} = h\otimes_{H^L}(k);
\end{aligned}$$

$$\begin{aligned}
\mathrm{can}(\mathrm{can}^{-1}(1_{(1)}\otimes h1_{(2)})) &= \mathrm{can}(1_{(1)}S(h_{(1)}1_{(2)}\otimes_{H^L} h_{(2)}1_{(3)})\\
&= 1_{(1)}S(h_{(1)}1_{(2)})h_{(2)}1_{(3)}\otimes h_{(3)}1_{(4)}\\
&= 1_{(1)}1_{(1')}\varepsilon(h_{(1)}1_{(2)}1_{(2')})\otimes h_{(2)}1_{(3)}\\
&= 1_{(1)}1_{(1')}\varepsilon(h_{(1)}1_{(3)})\varepsilon(1_{(2)}1_{(2')})\otimes h_{(2)}1_{(4)}\\
&= 1_{(1)}1_{(1')}\varepsilon(1_{(2)}1_{(2')})\otimes \varepsilon(h_{(1)}1_{(3)})h_{(2)}1_{(4)}\\
&= 1_{(1)}1_{(1')}\varepsilon(1_{(2)}1_{(2')})\otimes \varepsilon h1_{(3)}) = 1_{(1)}\otimes h1_{(2)}.
\end{aligned}$$

□

3. Groupoid gradings

Recall that a groupoid G is a category in which every morphism is an isomorphism. In this Section, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of G will be denoted by G_0, and the set of morphisms by G_1. The identity morphism on $x\in G_0$ will also be denoted by x. For $\sigma:\ x\to y$ in G_1, we write

$$s(\sigma)=x \text{ and } t(\sigma)=y,$$

respectively for the source and the target of σ. For every $x\in G$, $G_x=\{\sigma\in G\mid s(\sigma)=t(\sigma)=x\}$ is a group.
Let G be a groupoid, and k a commutative ring. The groupoid algebra is the direct product

$$kG=\bigoplus_{\sigma\in G_1} ku_\sigma,$$

with multiplication defined by the formula

$$u_\sigma u_\tau=\begin{cases} u_{\sigma\tau} & \text{if } t(\tau)=s(\sigma);\\ 0 & \text{if } t(\tau)\neq s(\sigma).\end{cases}$$

The unit element is $1=\sum_{x\in G_0}u_x$. kG is a weak Hopf algebra, with comultiplication, counit and antipode given by the formulas

$$\Delta(u_\sigma)=u_\sigma\otimes u_\sigma,\ \varepsilon(u_\sigma)=1 \text{ and } S(u_\sigma)=u_{\sigma^{-1}}.$$

Using the formula

$$\Delta(1)=\sum_{x\in G_0}u_x\otimes u_x.$$

We compute that $\Pi^L:\ kG\to kG$ is given by the formula

$$\Pi^L(u_\sigma)=\sum_{x\in G_0}\varepsilon(u_xu_\sigma)=u_{t(\sigma)},$$

hence

$$(kG)^L = \bigoplus_{x\in G_0} ku_x.$$

Let k be a commutative ring. A G-graded k-algebra is a k-algebra A together with a direct sum decomposition

$$A = \bigoplus_{\sigma\in G_1} A_\sigma,$$

such that

$$A_\sigma A_\tau \begin{cases} \subset A_{\sigma\tau} & \text{if } t(\tau) = s(\sigma); \\ = 0 & \text{if } t(\tau) \neq s(\sigma). \end{cases} \tag{30}$$

and

$$1_A \in \bigoplus_{x\in G_0} A_x. \tag{31}$$

Proposition 3.1. *Let G be a finite groupoid, and k a commutative ring. We have an isomorphism between the categories of kG-comodule algebras and G-graded k-algebras.*

Proof. Let (A,ρ) be a kG-comodule algebra, and define $\rho:\ A \to A\otimes kG$ by

$$A_\sigma = \{a\in A \mid \rho(a) = a\otimes u_\sigma\}.$$

From the fact that $A\otimes kG$ is a free left A-module with basis $\{u_\sigma \mid \sigma\in G_1\}$, it follows that $A_\sigma \cap A_\tau = \{0\}$ if $\sigma\neq\tau$.
For $a\in A$, we can write

$$\rho(a) = \sum_{\sigma\in G_1} a_\sigma\otimes u_\sigma.$$

From the coassociativity of ρ, it follows that

$$\sum_{\sigma\in G_1} \rho(a_\sigma)\otimes u_\sigma = \sum_{\sigma\in G_1} a_\sigma\otimes u_\sigma\otimes u_\sigma,$$

hence $\rho(a_\sigma)a_\sigma\otimes u_\sigma$, so $a_\sigma\in A_\sigma$ and

$$a = \sum_{\sigma\in G_1} a_\sigma\varepsilon(u_\sigma) = \sum_{\sigma\in G_1} a_\sigma \in_{\sigma\in G_1} A_\sigma.$$

If $a\in A_\sigma$ and $b\in A_\tau$, then $\rho(ab) = ab\otimes u_\sigma u_\tau$, and (30) follows.
(16) tells us that $\rho(1_A)\in A\otimes (kG)^L = \bigoplus_{x\in G_0} A\otimes u_x$, hence we can write

$$\rho(1_A) = \sum_{x\in G_0} 1_x\otimes u_x,$$

and

$$1_A = \sum_{x\in G_0} 1_x \in \bigoplus_{x\in G_0} A_x.$$

Conversely, let A be a G-graded algebra. For $a \in A$, let a_σ be the projection of A on A_σ. Then define $\rho(a) = \sum_{\sigma\in G_1} a_\sigma \otimes u_\sigma$. (16) then follows immediately from (31). □

Proposition 3.2. *Let A be a G-graded algebra, and take $x \in G_0$. Then $\bigoplus_{\sigma\in G_x} A_\sigma$ is a G_x-graded algebra, with unit 1_x. In particular, A_x is a k-algebra with unit 1_x.*

Proof. If $a \in A_\sigma$, with $t(\sigma) = x$, then

$$a \otimes \sigma = \rho(a) = \rho(1_A a) = \sum_{y\in G_0} 1_y a \otimes u_y u_\sigma = 1_x a \otimes \sigma,$$

hence $a = 1_x a$. In a similar way, we can prove that $a = a1_x$ if $s(a) = x$. The result then follows easily. □

Let A be a G-graded algebra. A G-graded right A-module is a right A-module together with a direct sum decomposition

$$M = \bigoplus_{\sigma\in G_1} M_\sigma$$

such that

$$M_\sigma A_\tau \begin{cases} \subset M_{\sigma\tau} & \text{if } t(\tau) = s(\sigma); \\ = 0 & \text{if } t(\tau) \neq s(\sigma). \end{cases}$$

A right A-linear map $f : M \to N$ between two G-graded right A-modules is called graded if $f(M_\sigma) \subset N_\sigma$, for all $\sigma \in G_1$. The category of G-graded right A-modules and graded A-linear maps is denoted by $\mathcal{M}_A^G$. The proof of the next result is then similar to the proof of Proposition 3.1.

Proposition 3.3. *Let G be a groupoid, and A a G-graded algebra. Then the categories $\mathcal{M}_A^G$ and $\mathcal{M}_A^{kg}$ are isomorphic.*

If $m \in M_\sigma$, with $s(\sigma) = x$, then

$$\rho(m) = m \otimes \sigma = \rho(m1) = \sum_{y\in G_0} m1_y \otimes \sigma x = m1_x \otimes \sigma,$$

hence $m1_x = m$.

Let $M \in \mathcal{M}_A^G \cong \mathcal{M}_A^{kg}$. Then $m \in M^{\mathrm{co}kG}$ if and only if

$$\rho(m) = \sum_{\sigma\in G_1} m_\sigma \otimes \sigma = \sum_{x\in G_0} m1_x \otimes x,$$

if and only if $m \in \bigoplus_{x\in G_0} M_x$. We conclude that

$$M^{\mathrm{co}kG} = \bigoplus_{x\in G_0} M_x. \tag{32}$$

In particular,

$$T = A^{\mathrm{co}kG} = \bigoplus_{x\in G_0} A_x.$$

If $N \in \mathcal{M}_T$, then $N = \bigoplus_{x\in G} N_x$, with $N_x = N1_x \in \mathcal{M}_{A_x}$. We have a pair of adjoint functors

$$F = - \otimes_T A: \ \mathcal{M}_T \to \mathcal{M}_A^G, \ G = (-)^{\mathrm{cok}G}: \ \mathcal{M}_A^G \to \mathcal{M}_T,$$

with unit and counit given by the formulas

$$\nu_N: \ N \to (N \otimes_T A)^{\mathrm{cok}G}, \ \nu_N(n) = \sum_{x\in G_0} n_x \otimes_T 1_x$$

$$\zeta_M: \ M^{\mathrm{cok}G} \otimes_T A \to M, \ \zeta_M(m \otimes_T a) = ma.$$

A G-graded k-algebra is called strongly graded if

$$A_\sigma A_\tau = A_{\sigma\tau} \text{ if } t(\tau) = s(\sigma).$$

Proposition 3.4. *Let A be a G-graded k-algebra. With notation as above, $\nu_N: \ N \to (N \otimes_T A)^{\mathrm{cok}G}$ is bijective, for every $N \in \mathcal{M}_T$.*

Proof. First observe that

$$(N \otimes_T A)_\sigma = N \otimes_T A_\sigma = N_{t(\sigma)} \otimes_T A_\sigma,$$

hence

$$(N \otimes_T A)^{\mathrm{cok}G} = \bigoplus_{x\in G_0} (N \otimes_T A)_x = \bigoplus_{x\in G_0} N_x \otimes_T A_x.$$

The map

$$f: \ (N \otimes_T A)^{\mathrm{cok}G} \to N, \ f(n_x \otimes a_x) = n_x a_x$$

is the inverse of ν_N: it is obvious that $f \circ \nu_N = N$. We also have that

$$(\nu_N \circ f)(n_x \otimes_T a_x) = n_x a_x \otimes_T 1_x = n_x \otimes_T a_x.$$

□

Let A be an algebra graded by a group G. It is a classical result from graded ring theory (see e.g. [16]) that A is strongly graded if and only if the categories of G-graded A-modules and A_1-modules are equivalent. This is also equivalent to A being a kG-Galois extension of A_e. We now present the groupoid version of this result.

Theorem 3.5. *Let G be a groupoid, and A a G-graded k-algebra. Then the following assertions are equivalent.*

1. *A is strongly graded;*
2. *(F, G) is a category equivalence;*
3. *$(A \otimes kG, \rho(1) = \sum_{x\in G_0} e_x \otimes x)$ is a Galois coring;*
4. $\mathrm{can}: \ A \otimes_T A \to A \otimes kG, \ \mathrm{can}(a \otimes b) = \sum_{\sigma\in G_1} ab_\sigma \otimes \sigma$ *is surjective.*

Proof. $\underline{1) \Rightarrow 2)}$. In view of Proposition 3.4, we only have to show that ζ_M is bijective, for every graded A-module M.
Take $m \in M_\sigma$, with $s(\sigma) = x$, $t(\sigma) = y$. Then $\sigma\sigma^{-1} = x$, and there exist

$a'_i \in A_{\sigma^{-1}}$, $a_i \in A_\sigma$ such that $\sum_i a'_i a_i = 1_x$. We have that $ma'_i \in M_\sigma A_{\sigma^{-1}} \subset M_y \subset M^{\mathrm{cok}G}$,and

$$\zeta_M(\sum_i ma'_i \otimes a_i) = \sum_i ma'_i a_i = m,$$

and it follows that ζ_M is surjective.
Take

$$m_j = \sum_{x\in G_0} m_{j,x} \in \bigoplus_{x\in G_0} M_x \text{ and } c_j \in A.$$

Assume

$$\zeta_M(\sum_j m_j \otimes_T c_j) = \sum_j m_j c_j = 0,$$

and take $\sigma \in G_1$ with $s(\sigma) = x$ and $t(\sigma) = y$. Then

$$0 = (\sum_j m_j c_j)_\sigma = \sum_j m_j c_{j,\sigma},$$

hence

$$\sum_j m_j \otimes_T c_{j,\sigma} = \sum_{i,j} m_j \otimes_T c_{j,\sigma} a'_i a_i == \sum_{i,j} m_j c_{j,\sigma} a'_i \otimes_T a_i = 0,$$

and

$$\sum_j m_j \otimes_T c_j = \sum_{\sigma\in G_1} \sum_j m_j \otimes_T c_{j,\sigma} = 0,$$

and it follows that ζ_M is injective.
<u>2) $\Rightarrow$ 3)</u> follows from the observation made before Theorem 1.1 (see [8, Prop. 3.1]).
<u>3) $\Rightarrow$ 4)</u> is trivial.
<u>4) $\Rightarrow$ 1)</u>. Take $\sigma, \tau \in G_1$ such that $s(\sigma) = t(\tau)$, and $c \in A_{\sigma\tau}$. Since can is surjective, there exist homgeneous $a_i, b_i \in A$ such that

$$\mathrm{can}(\sum_i a_i \otimes b_i) = \sum_{i,j} a_i b_j \otimes \deg(b_j) = c \otimes \tau.$$

On the left hand side, we can delete all the terms for which $\deg(b_j) \neq \tau$. So we find

$$\sum_{i,j} a_i b_j \otimes \tau = c \otimes \tau,$$

and

$$\sum_{i,j} a_i b_j = c.$$

On the left hand side, we can now delete all terms of degree different from $\sigma\tau$, since the degree of the right hand side is $\sigma\tau$. This means that we can delete all terms for which $\deg(a_i) \neq \sigma$. So we find that $\sum_{i,j} a_i b_j = c$, with $\deg(a_i) = \sigma$, and $\deg(b_i) = \tau$, and $A_\sigma A_\tau = A_{\sigma\tau}$. □

4. Groupoid actions

Let G be a groupoid, as in Section 3, with the additional assumption that G_1 is finite. Then kG is free of finite rank as a k-module, hence $Gk = (kG)^*$ is also a weak Hopf algebra. As a k-module,

$$Gk = \bigoplus_{\sigma \in G_1} kv_\sigma,$$

with

$$\langle v_\sigma, \tau \rangle = \delta_{\sigma,\tau}.$$

The algebra structure is given by the formulas

$$v_\sigma v_\tau = \delta_{\sigma,\tau} v_\sigma \ ; \ 1 = \sum_{\sigma \in G_1} v_\sigma,$$

and the coalgebra structure is

$$\Delta(v_\sigma) = \sum_{\tau\rho=\sigma} v_\tau \otimes v_r ho = \sum_{t(\tau)=t(\sigma)} v_\tau \otimes v_{\tau^{-1}\sigma},$$

$$\varepsilon(\sum_{\sigma \in G_1} a_\sigma v_\sigma) = \sum_{x \in G_0} a_x v_x.$$

The antipode is given by $S(v_\sigma) = v_{\sigma^{-1}}$. Observe that

$$\Delta(1) = 1_{(1)} \otimes 1_{(2)} = \sum_{t(\rho)=s(\sigma)} v_\tau \otimes v_\sigma.$$

Let A be a right Gk-comodule algebra. Then A is a left kG-module algebra, with left kG-action given by

$$\sigma \cdot a = \langle a_{[1]}, \sigma \rangle a_{[0]}.$$

The Gk-coaction ρ can be recovered from the action, using the formula

$$\rho(a) = \sum_{\sigma \in G_1} \sigma \cdot a \otimes v_\sigma. \tag{33}$$

In [9, Prop. 4.15], equivalent definitions of an H-module algebra are given. If we apply them in the case where $H = kG$, we find that a k-algebra with a left kG-module structure is a left kG-module algebra if

$$\sigma \cdot (ab) = (\sigma \cdot a)(\sigma \cdot b). \tag{34}$$

for all $\sigma \in G_1$ and $a, b \in A$, and the following equivalent conditions are satisfied.

$$\sigma \cdot 1_A = t(\sigma) \cdot 1_A \tag{35}$$

$$a(\sigma \cdot 1_A) = t(\sigma) \cdot a \tag{36}$$

$$(\sigma \cdot 1_A)a = t(\sigma) \cdot a \tag{37}$$

We will call A a left G-module algebra.

Proposition 4.1. *Let A be a left G-module algebra. Then $\{x\cdot 1_A \mid x\in G_0\}$ is a set of central orthogonal idempotents in A. Hence we can write*

$$A=\bigoplus_{x\in G_0} A_x,$$

where $A_\sigma=(\sigma\cdot 1_A)A=(t(\sigma)\cdot 1_A)=A_{t(\sigma)}$.

Proof. It follows from (33), with $a=b=1_A$ and $\sigma=x$, that $x\cdot 1_A$ is idempotent. From (36) and (37), it follows that $x\cdot 1_A$ is central. If $x\neq y\in G_0$, then we find, using (36), that $(x\cdot 1_A)(y\cdot 1_A)=y\cdot(x\cdot 1_A)=yx\cdot 1_A=0$. Finally

$$\sum_{x\in G_0} x\cdot 1_A=(\sum_{x\in G_0} x)\cdot 1_A=1_{kG}\cdot 1_A=1_A.$$

Observe also that $\sigma\cdot a=(\sigma\cdot a)(\sigma\cdot 1_A)\in A_\sigma$. □

We have

$$A\otimes Gk=\bigoplus_{\sigma\in G_1} Av_\sigma.$$

Let us compute the projection

$$g:\ \bigoplus_{\sigma\in G_1} Av_\sigma\to\bigoplus_{\sigma\in G_1} Av_\sigma$$

introduced in Section 2.

$$g(av_\sigma)=\sum_{\tau\in G_1} a(\tau\cdot 1_A)\otimes v_\sigma v_\tau=a(\sigma\cdot 1_A)\otimes v_\sigma.$$

We have seen in Lemma 2.1 that we have an A-coring

$$\mathcal{C}=\mathrm{Im}\,(g)=\bigoplus_{\sigma\in G_1} A_\sigma v_\sigma.$$

The right A-module structure is given by the formula

$$((\sigma\cdot 1_A)v_\sigma)a=(\sigma\cdot a)v_\sigma.$$

Now assume that $M\in\mathcal{M}^{\mathcal{C}}$, or, equivalently, $M\in\mathcal{M}_A^{Gk}$ (see Proposition 2.2). Then M is a right A-module, and also a right Gk-comodule, and a fortiori a left kG-module. Condition (22) is then equivalent to

$$\sum_{\sigma\in G_1}\sigma\cdot(ma)\otimes v_\sigma=\sum_{\sigma,\tau\in G_1}(\sigma\cdot m)(\tau\cdot a)\otimes v_\sigma v_\tau=\sum_{\sigma\in G_1}(\sigma\cdot m)(\sigma\cdot a)\otimes v_\sigma,$$

or

$$\sigma\cdot(ma)=(\sigma\cdot m)(\sigma\cdot a), \tag{38}$$

for all $\sigma\in G_1$, $m\in M$ and $a\in A$. We will also say that G acts as a groupoid of right A-semilinear automorphisms on M. Now $m\in M^G=M^{\mathrm{co}\mathcal{C}}$ if and only if

$$\rho(m)=\sum_{\sigma\in G_1}\sigma\cdot m\otimes v_\sigma$$

equals

$$m1_{[0]} \otimes 1_{[1]} = \sum_{\sigma \in G_1} m(\sigma \cdot 1_A) \otimes v_\sigma,$$

hence

$$M^G = \{m \in M \mid \sigma \cdot m = m(\sigma \cdot 1_A),\ \text{for all } \sigma \in G_1\}.$$

In particular,

$$T = A^G = \{a \in A \mid \sigma \cdot a = t(\sigma) \cdot a,\ \text{for all } \sigma \in G_1\}.$$

Let $B \to T$ be a ring morphism. Let us compute

$$\mathrm{can}:\ A \otimes_B A \to \bigoplus_{\sigma \in G_1} A_\sigma v_\sigma.$$

$$\begin{aligned} \mathrm{can}(a \otimes b) &= \sum_{\sigma \in G_1} (a(\sigma \cdot 1_A) \otimes v_\sigma) b \\ &= \sum_{\sigma \in G_1} a(\sigma \cdot 1_A)(\sigma \cdot b) \otimes v_\sigma \\ &= \sum_{\sigma \in G_1} a(\sigma \cdot b) \otimes v_\sigma. \end{aligned}$$

Our next goal is to compute

$${}^*\mathcal{C} = \underline{\mathrm{Hom}}(Gk, A) = \{f:\ Gk \to A \mid 1_{[0]} f(v_\tau 1_{[1]}) = f(v_\tau),\ \text{for all } \tau \in G_1\}.$$

Proposition 4.2. *Let G be finite groupoid, and A a G-module algebra. Then*

$$\underline{\mathrm{Hom}}(Gk, A) = \bigoplus_{\sigma \in G_1} u_\sigma A_\sigma,$$

as a right A-modules. The left A-module structure on $\underline{\mathrm{Hom}}(Gk, A)$ is given by the formula

$$a u_\sigma(\sigma \cdot 1_A) = u_\sigma(\sigma \cdot a), \tag{39}$$

and the multiplication is given by

$$U_\sigma \# U_\tau = U_{\tau\sigma}, \tag{40}$$

where we denoted $U_\sigma = u_\sigma(\sigma \cdot 1_A)$. The unit element of $\underline{\mathrm{Hom}}(Gk, A)$ is

$$\sum_{x \in G_0} U_x.$$

Proof. Observe that $\mathrm{Hom}(Gk, A) \cong (Gk)^* \otimes A = kG \otimes A$, so that every $f:\ Gk \to A$ can be written as

$$f = \sum_{\sigma \in G_1} u_\sigma a_\sigma,$$

with $a_\sigma = f(v_\sigma)$. u_σ is then the projection onto the component v_σ of Gk. We find that $f \in \underline{\mathrm{Hom}}(Gk, A)$ if and only if $f(v_\tau) = a_\tau$ equals

$$1_{[0]} f(v_\tau 1_{[1]}) = \sum_{\sigma\in G_1} (\sigma \cdot 1_A) f(v_\tau v_\sigma) = (\tau \cdot 1_A) a_\tau,$$

or, equivalently, $a_\tau \in A_\tau$, for every $\tau \in G_1$. Write $U_\sigma = u_\sigma(\sigma \cdot 1_A)$, then we have

$$\underline{\mathrm{Hom}}(Gk, A) = \bigoplus_{\sigma\in G_1} U_\sigma A = \bigoplus_{\sigma\in G_1} U_\sigma A_\sigma.$$

Observe that $U_\sigma(v_\tau) = (\sigma \cdot 1_A)\delta_{\sigma,\tau}$.
The left A-action on $\underline{\mathrm{Hom}}(Gk, A)$ is computed using (28). We compute

$$\begin{aligned}(aU_\sigma)(v_\tau) &= a_{[0]} U_\sigma(v_\tau a_{[1]}) = \sum_{\lambda\in G_1} (\lambda \cdot a) U_\sigma(v_\tau v_\lambda) \\ &= (\tau \cdot a) U_\sigma(v_\tau) = (\tau \cdot a)(\sigma \cdot 1_A)\delta_{\sigma,\tau},\end{aligned}$$

hence $(aU_\sigma)(v_\tau) = 0$ if $\sigma \neq \tau$. If $\sigma = \tau$, then

$$(aU_\sigma)(v_\sigma) = (\sigma \cdot a)(\sigma \cdot 1_A) = (\sigma \cdot a) = (U_\sigma(\sigma \cdot a))(v_\sigma),$$

and we conclude that

$$aU_\sigma = U_\sigma(\sigma \cdot a).$$

Using (26), we compute

$$\begin{aligned}(U_\sigma \# U_\tau)(v_\lambda) &= \sum_{\mu\nu=\lambda} U_\sigma(v_\nu)_{[0]} U_\tau(v_\mu f(v_\nu)_{[1]}) \\ &= \sum_{\rho\in G_1} \sum_{\mu\nu=\lambda} \rho \cdot ((\sigma \cdot 1_A)\delta_{\sigma,\nu}) U_\tau(v_\mu v_\rho) \\ &= \sum_{\mu\nu=\lambda} \mu((\sigma \cdot 1_A)\delta_{\sigma,\nu})(\tau \cdot 1_A)\delta_{\tau,\mu}.\end{aligned}$$

If $\tau\sigma \neq \lambda$, then $(U_\sigma \# U_\tau)(v_\lambda) = 0$.
If $\tau\sigma = \lambda$, then

$$\begin{aligned}(U_\sigma \# U_\tau)(v_\lambda) &= (\tau \cdot (\sigma \cdot 1_A))(\tau \cdot 1_A) = (\tau\sigma \cdot 1_A)(\tau \cdot 1_A) \\ &= (t(\tau\sigma) \cdot 1_A)(t(\tau) \cdot 1_A) = (t(\tau\sigma) \cdot 1_A)(t(\tau\sigma) \cdot 1_A) \\ &= t(\tau\sigma) \cdot 1_A = (\tau\sigma) \cdot 1_A = U_{\tau\sigma}(v_\lambda),\end{aligned}$$

and we conclude that $U_\sigma \# U_\tau = U_{\tau\sigma}$. □

Recall that a ring morphism $A \to R$ is called *Frobenius* if there exists an A-bimodule map $\overline{\nu} : R \to A$ and $e = e^1 \otimes_A e^2 \in R \otimes_A R$ (summation implicitly understood) such that

$$re^1 \otimes_A e^2 = e^1 \otimes_A e^2 r \tag{41}$$

for all $r \in R$, and

$$\overline{\nu}(e^1)e^2 = e^1\overline{\nu}(e^2) = 1. \tag{42}$$

This is equivalent to the restrictions of scalars $\mathcal{M}_R \to \mathcal{M}_A$ being a Frobenius functor, which means that its left and right adjoints are isomorphic (see [11, Sec. 3.1 and 3.2]). $(e, \overline{\nu})$ is then called a Frobenius system.

Proposition 4.3. *Let G be finite groupoid, and A a G-module algebra. Then the ring morphism $A \to \underline{\mathrm{Hom}}(Gk, A) = {}^*\mathcal{C}$ is Frobenius.*

Proof. The Frobenius system is the following:

$$e = \sum_{\sigma\in G_1} U_{\sigma^{-1}} \otimes U_\sigma;$$

$$\overline{\nu}(\sum_{\sigma\in G_1})u_\sigma(\sigma\cdot 1_A)a_\sigma) = \sum_{x\in G_0}(x\cdot 1_A)a_x.$$

For all $a \in A$, we have that

$$\begin{aligned} ae &= \sum_{\sigma\in G_1} U_{\sigma^{-1}}(\sigma^{-1}\cdot a)\otimes U_\sigma = \sum_{\sigma\in G_1} U_{\sigma^{-1}}\otimes U_\sigma((\sigma\sigma^{-1})\cdot a) \\ &= \sum_{\sigma\in G_1} U_{\sigma^{-1}}\otimes U_\sigma(t(\sigma)\cdot a) = \sum_{\sigma\in G_1} U_{\sigma^{-1}}\otimes U_\sigma(\sigma\cdot 1_A)a = ea; \end{aligned}$$

and

$$\begin{aligned} \overline{\nu}(a\sum_{\sigma\in G_1})u_\sigma(\sigma\cdot 1_A)a_\sigma) = \overline{\nu}(\sum_{\sigma\in G_1})u_\sigma(\sigma\cdot a)a_\sigma) \\ &= \sum_{x\in G_0}(x\cdot a)a_x = \sum_{x\in G_0} a(x\cdot 1_A)a_x \\ &= a\overline{\nu}(\sum_{\sigma\in G_1})u_\sigma(\sigma\cdot 1_A)a_\sigma), \end{aligned}$$

and it follows that $\overline{\nu}$ is left A-linear. It is clear that $\overline{\nu}$ is right A-linear. Finally,

$$\begin{aligned} \overline{\nu}(e^1)e^2 &= \sum_{x\in G_0}(x\cdot 1_A)u_x(x\cdot 1_A) \\ &= \sum_{x\in G_0}(x\cdot 1_A)u_x(x^2\cdot 1_A)(x\cdot 1_A) = \sum_{x\in G_0} U_x = 1. \end{aligned}$$

In a similar way, we show that $e^1\overline{\nu}(e^2) = 1$. □

Using (27), we compute the map ${}^*\mathrm{can}:\ \underline{\mathrm{Hom}}(H, A) \to {}_B\mathrm{End}(A)$:

$${}^*\mathrm{can}(U_\sigma a_\sigma)(b) = (\sigma\cdot b)a_\sigma.$$

Let us now compute the module Q (see (29)). From [12, Theorem 2.7], it follows that Q and A are isomorphic as abelian groups. This will follow from our computations.

Proposition 4.4. *Let G be finite groupoid, and A a G-module algebra. Then*

$$Q = \{\sum_{\sigma\in G_1} u_\sigma(\sigma\cdot a) \mid a\in A\}.$$

Proof. Take $f=\sum_{\sigma\in G_1} U_\sigma a_\sigma\in \mathrm{Hom}(H,A)$. Then $f\in Q$ if and only if (29) is satisfied. Take $h=v_\tau$ in (29). The right hand side of (29) is

$$\sum_{\mu\in G_1} a_\tau(\mu\cdot 1_A)\otimes v_\mu=\sum_{\mu\in G_1}(\mu\cdot 1_A)a_\tau\otimes v_\mu.$$

The left hand side of (29) is

$$\sum_{\sigma\in G_1}\sum_{\mu\nu=\tau}\sum_{\rho\in G_1}\rho\cdot(U_\sigma a_\sigma)(v_\nu)\otimes v_\mu v_\rho=\sum_{\mu\nu=\tau}\mu\cdot a_\nu\otimes v_\mu.$$

It follows that $f\in Q$ if and only if

$$\sum_{\mu\in G_1}(\mu\cdot 1_A)a_\tau\otimes v_\mu=\sum_{\mu\nu=\tau}\mu\cdot a_\nu\otimes v_\mu, \tag{43}$$

for all $\tau\in G_1$. Assume that $f\in Q$, and take the v_τ component of (43). Then $\nu=s(\tau)$ and

$$(\tau\cdot 1_A)a_\tau=a\tau=\tau\cdot a_{s(\tau)}. \tag{44}$$

This means that f is completely determined by

$$a=\sum_{x\in G_0}a_x\in A=\bigoplus_{x\in G_0}A_x.$$

Conversely, take $a\in A$, let $a_x=a(x\cdot 1_A)\in A_x$, and define a_τ using (44). Then

$$(\tau\cdot 1_A)a_\tau=(\tau\cdot 1_A)(\tau\cdot a_{s(\tau)})=\tau\cdot(1_A s(\tau))a_\tau,$$

so $a_\tau\in A_\tau$, as needed. Also observe that

$$a_\tau=\tau\cdot a(s(\tau)\cdot 1_A))=(\tau\cdot a)((\tau s(\tau))\cdot 1_A)=(\tau\cdot a)(\tau\cdot 1_A)=\tau\cdot a.$$

We then claim that

$$f=\sum_{\sigma\in G_1}U_\sigma a_\sigma\in Q.$$

Take $\sigma,\tau\in G_1$ with $t(\sigma)\neq t(\tau)$. Then for all $a\in A$, we have

$$(\sigma\cdot 1_A)(\tau\cdot 1_A)a=(\sigma\cdot 1_A)(t(\tau)\cdot a)=(t(\sigma)t(\tau))\cdot a=0.$$

It follows that the left hand side of (43) amounts to

$$\sum_{t(\mu)=t(\tau)}(\mu\cdot 1_A)a_\tau\otimes v_\mu=\sum_{t(\mu)=t(\tau)}a_\tau\otimes v_\mu.$$

The right hand side of (43) is

$$\sum_{t(\mu)=t(\tau)}\mu\cdot a_{\mu^{-1}\tau}\otimes v_\mu.$$

If $t(\mu)=t(\tau)$, then

$$\mu\cdot a_{\mu^{-1}\tau}=\mu\cdot((\mu^{-1}\tau)\cdot a_{s(\mu^{-1}\tau)}=\tau\cdot a_{s(\tau)}=a_\tau,$$

and (43) follows. Hence $f\in Q$. $\square$

We have seen in Section 2 that Q is a $(\underline{\mathrm{Hom}}(Gk,A),T)$-bimodule. Using Proposition 4.4, we can transport this bimodule structure to A. We find the following bimodule structure on A:

$$(\sum_{\tau\in G_1} U_\tau b_\tau)\cdot a\cdot u=\sum_{\tau\in G_1}\tau^{-1}\cdot(b_\tau a)u.$$

A is also a $(T,\underline{\mathrm{Hom}}(Gk,A))$-bimodule. The structure is

$$u\bullet a\bullet(\sum_{\tau\in G_1} U_\tau b_\tau)=\sum_{\tau\in G_1} u(\tau\cdot a)b_\tau.$$

We have a Morita context $(T,\underline{\mathrm{Hom}}(Gk,A),A,Q,\tau,\mu)$, and, a fortiori, a Morita context $(T,\underline{\mathrm{Hom}}(Gk,A),A,A,\tau,\mu)$. The connecting maps

$$\tau:\ A\otimes_{\underline{\mathrm{Hom}}(Gk,A)}A\to T \text{ and } \mu:\ A\otimes_T A\to\underline{\mathrm{Hom}}(Gk,A)$$

are given by the formulas

$$\tau(b\otimes a)=\sum_{\sigma\in G_1}\sigma\cdot(ba);$$

$$\mu(a\otimes b)=\sum_{\sigma\in G_1}U_\sigma(\sigma\cdot a)b.$$

It follows from Proposition 1.2 that τ is surjective if and only if there exists $a\in A$ such that $\sum_{\sigma\in G_1}\sigma\cdot a=1$. Theorem 2.6 takes the following form.

Theorem 4.5. *Let G be a finite groupoid, and A a G-module algebra. Take a subring B of $T=A^G$. Then, with notation as above, the following assertions are equivalent.*

(1)
- can *is an isomorphism;*
- *A is faithfully flat as a left B-module.*

(2)
- *can *is an isomorphism;*
- *A is a left B-progenerator.*

(3)
- *$B=T$;*
- *the Morita context $(B,\underline{\mathrm{Hom}}(Gk,A),A,A,\tau,\mu)$ is strict.*

(4)
- *$B=T$;*
- *(F,G) is an equivalence of categories.*

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